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R-Local Delaunay Inhibition Model

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Abstract Unlike in the classical framework of Gibbs point processes (usually acting on the complete graph), in the context of nearest-neighbour Gibbs point processes the nonnegativeness of the interaction functions do not ensure the local stability property. This paper introduces domain-wise (but not pointwise) inhibition stationary Gibbs models based on some tailor-made Delaunay subgraphs. All of them are subgraphs of the R-local Delaunay graph, defined as the Delaunay subgraph specifically not containing the edges of Delaunay triangles with circumscribed circles of radii greater than some large positive real value R. The usual relative compactness criterion for point processes needed for the existence result is directly derived from the Ruelle-bound of the correlation functions. Furthermore, assuming only the nonnegativeness of the energy function, we have managed to prove the existence of the existence of R-local Delaunay stationary Gibbs states based on nonnegative interaction functions thanks to the use of the compactness of sublevel sets of the relative entropy.

Keywords Stationary Gibbs states · Inhibition property · Delaunay triangulation · D.L.R. equations · Local specifications · Correlation functions

The existence result of stationary Gibbs states relies on a relative compactness assumption which may be expressed in different ways. In the classical framework of Gibbs point processes, Ruelle [14, 15] has introduced the class of Gibbs point processes based on a superstable lower regular finite energy function. The usual relative compactness criterion for point processes (see Proposition 9.1.V of [6]) is directly obtained by bounding the correlation functions. Before this work, the models considered were often based on hard-core or inhibition interactions. They directly inherit the stability of the local energy function which is a sufficient condition of the Ruelle-bound of the correlation functions. In the context of

In memory of Etienne Bertin.

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nearest-neighbour Gibbs point processes, introduced by Baddeley and Møller [1], the hard-core model is well-defined since usual hard-core interactions are intrinsically defined on the well-known first nearest neighbour graph. Surprisingly, the inhibition model is not directly adaptable. Bertin, Billiot and Drouilhet [2, 3] have first tried to prove the existence of point processes with interactions based on the Delaunay graph. They have reached their aim by introducing a Delaunay subgraph slightly modified to provide the local stability property. This subgraph is tailor-made to have the maximum number of neighbours of any finite configuration of points upperbounded by some finite constant. It is noticeable that the authors do not consider the inhibition model, because unlike in the classical framework, the nonnegativeness of the interaction function does not imply the local stability property.

The inhibition property generally expresses the notion of repulsion between particles via the nonnegativeness of the energy or interaction functions. In a more general case than the classical Gibbs point processes framework where interactions are based on some graph different from the complete graph, the kind of the repulsion differs depending on the form of the considered energy functions. The notions of pointwise inhibition, domain-wise inhibition and global inhibition are respectively associated with the pointwise local energy, the domain-wise local energy and the finite energy. Unlike the complete graph, a nearest-neighbour graph of any configuration of points does not contain that of any subconfiguration. As a direct consequence, the global inhibition occurs when the pointwise inhibition does not. This behavior does not make sense for the complete graph since the nonnegativeness of the interaction functions directly implies that of the three forms of energy functions.

First and foremost, this paper aims at introducing stationary Gibbs point processes based on some slightly modified Delaunay subgraphs which satisfy the domain-wise inhibition property providing the Ruelle-bound of the correlation functions. A further requirement is that these Delaunay subgraphs behave on a small scale exactly as the original Delaunay graph in order to easily inherit percolation properties, possibly useful in the extension of some results in the phase transition phenomenon (see [4]). More precisely, on the one hand, we want this subgraph to be different from the Delaunay graph only for edges between points "close" to some great enough empty space at least containing a ball of radius R. On the other hand, in order to remove the "negative" contribution in the expression of the domain-wise local energy, the graph of any configuration φ of points has to include the graph of the subconfiguration $\varphi \cap \widetilde{\Lambda}^c$ at least on each domain $\widetilde{\Lambda}$ of some cofinal set of Borel sets. The resulting Delaunay subgraphs are all subgraphs of the R-local Delaunay graph defined as the Delaunay graph in which the edges of the triangles with circumscribed circles of radii greater than some fixed value R have been removed. As a direct consequence of the local property of the domain-wise local energies and the Ruelle-bound of the correlation functions, the set of stationary Gibbs states is then nonempty.

Furthermore, let us underline that these subgraphs are very slight modifications of the R-local Delaunay graph. In particular, as the value of R increases, the subgraphs come closer to the R-local Delaunay graph. Then arises the question of why the existence of Gibbs models with nonnegative interaction functions acting on the R-local Delaunay graph has not yet been proved. Since this model clearly does not inherit the domain-wise inhibition property, the Ruelle-bound of the correlation functions will most likely not easily be obtained. One may wonder whether the weakest inhibition property, i.e. global inhibition, will be enough to ensure the compactness criterion needed for the existence of stationary Gibbs states. By using the tools proposed in Georgii-Häggström [8] based on the compactness of sublevel sets of the relative entropy density [7], we have finally succeeded in proving it. Let us mention that these tools are obviously also directly applicable to the previous models.

In the present paper, we investigate two approaches to obtain the compactness criterion. The first classical approach aims at achieving the Ruelle-bound of the correlation functions which directly implies the usual compactness criterion for point processes. Before this work, our belief was that this property was the minimal requirement to ensure the existence of stationary Gibbs states. The second approach uses the tools proposed in Georgii-Häggström [8] and is based on the compactness of sublevel sets of the relative entropy density [7]. These tools are so powerful that they allow us to directly deal with a nonnegative finite energy function based on the R-local Delaunay graph. In the classical framework of superstable lower regular point processes, let us notice that, as mentioned by the authors in [8], this approach only proposes a new way to prove the already well-known result of the existence of stationary Gibbs states first obtained by Ruelle [15] with the first approach. In order to appreciate the difference between the two approaches, it is interesting to compare both sufficient assumptions of the compactness criteria when expressed in terms of energy. Let us introduce the domain-wise local energy $V(\varphi \cap \Lambda|\varphi^o \cap \Lambda^c)$ defined for any Borel bounded set Λ as the energy required to insert the finite subconfiguration of φ of points in Λ inside the subconfiguration of φ^o of points in Λ^c. It is well-known that the Ruelle-bound of the correlation functions is directly implied by the domain-wise local stability: there exists $K \geq 0$ such that for all configuration φ^o, $V(\varphi \cap \Lambda|\varphi^o \cap \Lambda^c) \geq -K\#\varphi_\Lambda$. The compactness of sublevel sets of the relative entropy density is obtained whenever there exists some outside configuration φ^o and $K_0 \geq 0$ such that $V(\varphi \cap \Lambda|\varphi^o \cap \Lambda^c) \geq -K_0|\Lambda|$ where $|\Lambda|$ denotes the size of Λ. In the particular case of the free boundary (i.e. $\varphi^o = \emptyset$), the local energy function in the last assumption could be simply replaced with the finite energy function. The difficulty to obtain these assumptions widely differs depending on the point process framework. Indeed, in the case of classical point processes the nonnegativeness of the interaction functions directly implies both assumptions whereas in the case of nearest neighbour point processes only the second assumption remains true. Our previous belief that the Ruelle-bound of the correlation functions was the minimal requirement to inherit the compactness criterion needed for the existence result is now outdated. Note, however, that the Ruelle-bound of the correlation functions provides stronger information on the moments of the limiting stationary Gibbs measures.

We hope that the nearest-neighbour continuum models are interesting for low temperatures (not too low for a classical approach) as an alternative to standard models on regular networks, because they allow more degrees of freedom. This could provide new applications in crystallography, especially for the study of the rigidity and the plasticity properties of glasses or those of ferromagnetic fluids or liquid crystals (smectic A, C, nematic N). See for example [5, 9] and the references therein. In particular, it seems that emptiness is important for the study of the equilibrium tension in a membrane [5, 12]. More generally, it is well-known that the Voronoi graph, and its Voronoi regions (rather called Wigner-Seitz grid and Brillouin zones in the framework of physics), are essential to understand electrical currents, wave propagation and phase transition.

After giving some notations and preliminaries about the R-local Delaunay graph and the $\star R$-local Delaunay graphs in Sect. 1, we introduce the different inhibition Gibbs models based on finite energies with interaction on these graphs in Sect. 2. After introducing the definition of the $\widetilde{\mathcal{B}}$-cofinally local stability with $\widetilde{\mathcal{B}}$ a cofinal subset of the set of the Borel sets, we establish the nonemptiness of the set of stationary Gibbs states associated with each of these related energy functions in Sect. 3.

1 Preliminaries

In this section and for the rest of the paper, R designates some fixed nonnegative real number. For convenience, we use the following convention: a formula using the symbol ":="

introduces some new notation in the left hand side and its definition in the right hand side. For any given Borel set $\Lambda \subset \mathbb{R}^d$ with $d \geq 2$, one denotes by Ω and Ω_Λ the classes of locally finite subsets of points, called configurations in this paper, in $\mathbb{R}^d$ and Λ respectively. In particular, Ω_f denotes the set of finite configurations in Ω. Moreover, for any set Δ (not necessarily a Borel set) $\mathcal{P}_2(\Delta)$ (resp. $\mathcal{P}(\Delta)$) designates the set of pairs (resp. subsets) of elements in Δ. Let $\mathcal{B}$ and $\mathcal{B}_b$ be the set of Borel sets and bounded Borel sets of $\mathbb{R}^d$. An element φ of Ω could be represented as $\varphi = \sum_{i \in \mathbb{N}} \delta_{x_i}$ which is a simple counting Radon measure in $\mathbb{R}^d$ (i.e. all the points x_i of $\mathbb{R}^d$ are distinct) where $\forall \Lambda \in \mathcal{B}$, $\delta_x(\Lambda) := 1_\Lambda(x)$ is the Dirac measure and $1_A(\cdot)$ is the indicator function of a set A. For convenience, φ_Δ simply denotes $\varphi \cap \Delta$ where Δ is any set. The number of points of φ inside some set $\Delta \in \mathcal{B}$ denoted by $N_\Delta(\varphi)$ is also expressed by $\#(\varphi_\Delta)$ and $\varphi(\Delta)$. The space Ω is equipped with the vague topology, that is, the weak topology for Radon measures with respect to the set of continuous functions vanishing outside a compact set. $\mathcal{F}$ is the σ-field spanned by the maps $\varphi \longrightarrow \varphi(\Delta)$, $\Delta \in \mathcal{B}_b$, $\forall \varphi \in \Omega$. The corresponding σ-field $\mathcal{F}_\Lambda$ is similarly defined on Ω_Λ. Furthermore, for any $\Lambda \in \mathcal{B}_b$,

$$(\Omega, \mathcal{F}) = (\Omega_\Lambda, \mathcal{F}_\Lambda) \times (\Omega_{\Lambda^c}, \mathcal{F}_{\Lambda^c})$$

where Λ^c denotes the complementary of Λ in $\mathbb{R}^d$. Let $\widetilde{\mathcal{F}}_\Lambda$ be the reverse projection of $\mathcal{F}_\Lambda$ under the previous identification, so that $\widetilde{\mathcal{F}}_\Lambda$ is a σ-field on Ω.

A point process Φ on $\mathbb{R}^d$ (respectively Φ_Λ on Λ) is a random variable on Ω (respectively on Ω_Λ) and is associated with a probability distribution P on $(\Omega, \mathcal{F})$, (respectively P_Λ on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$).

Some configuration φ is said to be in general position when no $d+2$ points lie on the same hypersphere (with no point inside) and no $l+1$ ($l = 2, \ldots, d$) points lie on some $l-1$ dimensional affine subspace of $\mathbb{R}^d$. For any simplex ψ (triangle when $d = 2$) in some configuration φ, one denotes by $C(\psi)$ the greatest hypersphere circumscribing ψ with no point of φ inside its interior. The radius and the center (Voronoi vertice) of such a hypersphere are respectively denoted by $r(\psi)$ and $c(\psi)$. One notices that, for any simplex ψ, $\#(C(\psi) \cap \varphi) = d+1$ holds only if the configuration φ is in general position.

For any set A, its complement in $\mathbb{R}^d$, its interior, its closure and its boundary are respectively denoted by A^c, $\mathring{A}$, $\overline{A}$ and ∂A. One introduces the set $A \oplus R := \bigcup_{x \in A} B(x, R)$ where $B(x, R) := \{y \in \mathbb{R}^d : \|x - y\| \leq R\}$ is the usual ball of radius R and $\|x - y\|$ designates the Euclidean distance between the points x and y in $\mathbb{R}^d$. Moreover, for any $x \in \mathbb{R}^d$, ϑ_x denotes the translation operator defined by $\vartheta_x A := \{y + x : y \in A\}$.

For some family $G = \{G(\varphi)\}_{\varphi \in \Omega}$ of graph $G(\varphi) \subset \mathcal{P}_2(\varphi)$ related to each configuration φ, one denotes by $\mathcal{N}_G(\psi|\varphi) := \{x \in \varphi : \exists x' \in \psi, \{x, x'\} \in G\}$ the set of neighbours of the subconfiguration ψ of φ. For convenience, let us introduce $\overline{\mathcal{N}_G}(\psi|\varphi) := \mathcal{N}_G(\psi|\varphi) \cup \psi$. When inserting the points of some finite configuration φ_1 into some finite configuration φ_2 such that $\varphi_1 \cap \varphi_2 = \emptyset$, the set of residual edges $G(\varphi_1|\varphi_2)$ is decomposed as follows

$$G(\varphi_1|\varphi_2) := \underbrace{(G(\varphi_1 \cup \varphi_2) \setminus G(\varphi_2))}_{G^+(\varphi_1|\varphi_2)} \cup \underbrace{(G(\varphi_2) \setminus G(\varphi_1 \cup \varphi_2))}_{G^-(\varphi_1|\varphi_2)}$$

where $G^+(\varphi_1|\varphi_2)$ and $G^-(\varphi_1|\varphi_2)$ are respectively the sets of created and deleted edges.

1.1 The Delaunay Graph

Before defining the R-local Delaunay graph we first need to recall the definition of the Delaunay graph.

Definition 1 For some φ in Ω in general position, one defines $Del_{d+1}(\varphi)$ as the unique decomposition into simplexes ψ in which the convex hull of the hypersphere $C(\psi)$ does not contain any point of $\varphi \setminus \psi$. The Delaunay graph is then defined by the set of edges:

$$Del_2(\varphi) := \bigcup_{\psi \in Del_{d+1}(\varphi)} \mathcal{P}_2(\psi). \tag{1}$$

According to the previous definition, one can assert that in the two dimensional case, the Delaunay graph is a triangulation whenever the configuration φ is in general position. The Delaunay graph is useful to define nearest neighbour interactions because it is related to the Voronoi tessellation. For each point x of some configuration φ in general position, let us introduce its Voronoi cell $Vor(x|\varphi) := \{x' \in \mathbb{R}^d : \|x' - x\| \leq \|x' - x''\|, \forall x'' \in \varphi \setminus \{x\}\}$ representing the subset of points in $\mathbb{R}^d$ nearer to x than the other points of the configuration φ. Two distinct points x and x' in φ are neighbours in the Delaunay graph whenever their Voronoi cells intersect. The Delaunay edge is then graphically represented by the corresponding intersection $Vor(x|\varphi) \cap Vor(x'|\varphi)$.

Let us recall the local property of the Delaunay graph expressed by the following relation: for any x in some configuration φ:

$$\begin{aligned}
Del_2(\varphi) \cap \mathcal{P}_2\left(\overline{\mathcal{N}}_{Del_2}(\{x\}|\varphi)\right) &= Del_2(\varphi) \cap Del_2(\overline{\mathcal{N}}_{Del_2}(\{x\}|\varphi)) \\
&\subset Del_2(\overline{\mathcal{N}}_{Del_2}(\{x\}|\varphi)) \\
Del_2(\varphi \setminus \{x\}) \cap \mathcal{P}_2\left(\mathcal{N}_{Del_2}(\{x\}|\varphi)\right) &= Del_2(\varphi \setminus \{x\}) \cap Del_2(\mathcal{N}_{Del_2}(\{x\}|\varphi)) \\
&\subset Del_2(\mathcal{N}_{Del_2}(\{x\}|\varphi)).
\end{aligned}$$

As a consequence of this property, one obtains for any bounded Borel set Δ

$$Del_2^+(\varphi_\Delta|\varphi_{\Delta^c}) = Del_2^+(\varphi_\Delta|\mathcal{N}_{Del_2}(\varphi_\Delta|\varphi) \cap \Delta^c)$$

and

$$Del_2^-(\varphi_\Delta|\varphi_{\Delta^c}) = Del_2^-(\varphi_\Delta|\mathcal{N}_{Del_2}(\varphi_\Delta|\varphi) \cap \Delta^c)$$

which directly leads to

$$Del_2(\varphi_\Delta|\varphi_{\Delta^c}) = Del_2(\varphi_\Delta|\mathcal{N}_{Del_2}(\varphi_\Delta|\varphi) \cap \Delta^c) \subset \mathcal{P}_2\left(\overline{\mathcal{N}}_{Del_2}(\varphi_\Delta|\varphi)\right). \tag{2}$$

Only the knowledge of the points of $\overline{\mathcal{N}}_{Del_2}(\varphi_\Delta|\varphi)$ is needed to update the Delaunay graph of $\varphi = \varphi_\Delta \cup \varphi_{\Delta^c}$ from the given Delaunay graph of φ_{Δ^c}.

The Delaunay edges are generally defined after the introduction of the set of the Delaunay $d+1$-simplex as in (1). The following proposition provides a direct characterization of the Delaunay graph with no need to define the Delaunay $d+1$-simplex.

Proposition 1

$$\xi \in Del_2(\varphi) \Longleftrightarrow \mathcal{Z}_\varphi(\xi) \neq \emptyset$$

where $\mathcal{Z}_\varphi(\xi) := \{(c,r) \in \mathbb{R}^d \times \mathbb{R}^+ : \xi \subset \partial B(c,r)$ *and* $\mathring{B}(c,r) \cap \varphi = \emptyset\}$.

Proof $\mathcal{Z}_\varphi(\xi)$ nonempty means that there exists some ball $B(c,r)$ circumscribing a d-simplex ψ containing ξ and such that $\mathring{B}(c,r) \cap \varphi = \emptyset$. □

For any edge $\xi \in Del_2(\varphi)$ one may define its **influence region** $Z_\varphi(\xi)$ as the intersection of the balls characterized by the nonempty set $\mathcal{Z}_\varphi(\xi)$:

$$Z_\varphi(\xi) := \bigcap_{(c,r)\in\mathcal{Z}_\varphi(\xi)} B(c,r) = \bigcap_{\psi\in Del_{d+1}(\varphi):\psi\supset\xi} B(c(\psi), r(\psi))$$

Each point z of $Z_\varphi(\xi)$, if inserted in φ, would lead to the deletion of the edge ξ, that is: $\xi \in Del_2^-(\{z\}|\varphi)$.

1.2 The R-Local Delaunay Subgraphs

In the nearest neighbour point process framework, one may find irrelevant to consider interactions between Delaunay neighbours too far from each other. In this section, we will introduce some Delaunay subgraphs in which no edges are greater than some fixed distance $2R$. The most obvious one is the R-local Delaunay graph defined by:

$$Del_{2,R}^{\bigcirc}(\varphi) := \bigcup_{\psi\in Del_{d+1,R}^{\bigcirc}(\varphi)} \mathcal{P}_2(\psi). \tag{3}$$

where $Del_{d+1,R}^{\bigcirc}(\varphi) := \{\psi \in Del_{d+1}(\varphi) : r(\psi) \le R\}$.

In the rest of this paper, this Delaunay subgraph is referred to as the $\bigcirc R$-local Delaunay graph. The characteristic properties of the R-local Delaunay graph are given below.

Proposition 2 *For any configurations* φ, φ_1 *and* φ_2:

1. $Del_{2,R}^{\bigcirc}(\varphi)$ *does not contain any edge of length greater than* $2R$.
2. *For any* $\xi \in Del_{2,R}^{\bigcirc}(\varphi)$, $diam(Z_\varphi(\xi)) \le 2R$ *where* $diam(\Delta) := \sup_{(z_1,z_2)\in\Delta^2} \|z_1 - z_2\|$ *is the diameter of a set* Δ.
3. *Whenever* $\inf_{x_1\in\varphi_1, x_2\in\varphi_2} \|x_1 - x_2\| > 2R$,

$$Del_{2,R}^{\bigcirc}(\varphi_1 \cup \varphi_2) = Del_{2,R}^{\bigcirc}(\varphi_1) \cup Del_{2,R}^{\bigcirc}(\varphi_2).$$

Proof The last two assertions are direct consequences of the first one. An edge ξ of length at least $2R$ does not belong to $Del_{2,R}^{\bigcirc}(\varphi)$ since all $\psi \in Del_{d+1}(\varphi)$ containing ξ has a radius necessarily greater than R and therefore cannot belong to $Del_{d+1,R}^{\bigcirc}(\varphi)$. □

One of the goals of this paper is to introduce domain-wise inhibition Delaunay models. We are then interested in the construction of Delaunay subgraphs for which the graph of any configuration φ contains the subgraph of every subconfiguration φ_{Λ^c} with Λ large enough. The R-local Delaunay graph clearly does not satisfy this property and cannot be considered as a candidate for a domain-wise inhibition Delaunay model.

On a small scale, we would also expect these Delaunay subgraphs to be local like the original Delaunay graph (and then the R-local Delaunay graph). Let us notice that the Delaunay subgraphs introduced in [2, 3] unfortunately does not comply with this requirement. Indeed, in order to bound the maximum number of neighbours uniformly on all the configurations of points, the authors only kept the Delaunay triangles whose smallest angles were great enough.

We will now introduce new Delaunay subgraphs which are R-local Delaunay subgraphs and that inherit at least the properties given in the previous proposition. Let us denote by respectively $\mathcal{D}_R^{\oplus}$, $\mathcal{D}_R^{\boxtimes}$ and $\mathcal{D}_R^{\boxplus}$ the set of balls of diameter $2R$, the set of hypercubes of diameter $2R$ and the set of hypercubes of diameter $2R$ of the form $\prod_{i=1}^{d}[a_i, b_i]$. For convenience in this paper, the notation $\star$ standing for $\oplus$, $\boxplus$ and $\boxtimes$ will be used to simultaneously consider the three associated R-local Delaunay subgraphs. In addition, the notation $\bullet$ standing for $\bigcirc$ and $\star$ will be used to describe general statements applying to the R-local Delaunay graph and its three R-local Delaunay subgraphs.

In particular, $\mathcal{B}_R^{\star}$ denotes the set of all the unions of elements of $\mathcal{D}_R^{\star}$:

$$\mathcal{B}_R^{\star} := \left\{ \Lambda \in \mathcal{B} : \exists \mathcal{D} \subset \mathcal{D}_R^{\star},\ \Lambda = \bigcup_{\Delta \in \mathcal{D}} \Delta \right\}.$$

We then propose to introduce the definition of $\star R$-vacuum of φ which is a particular element of $\mathcal{B}_R^{\star}$ representing the subset of $\mathbb{R}^d$ decomposed into the union of connex components of $\mathcal{D}_R^{\star}$ (of diameter $2R$) with its interior not intersecting points of φ.

Definition 2 For any $\varphi \in \Omega$, one defines the $\star R$-vacuum of φ as follows:

$$\emptyset_R^{\star}(\varphi) := \bigcup_{\substack{\Delta \in \mathcal{D}_R^{\star}: \\ \mathring{\Delta} \cap \varphi = \emptyset \\ \partial \Delta \cap \varphi \neq \emptyset}} \Delta.$$

In particular, one may notice that $\emptyset_R^{\oplus}(\varphi) = (\varphi \oplus R)^c \oplus R$. The respective definitions of the $\star R$-local Delaunay graphs are then proposed.

Definition 3 The $\star R$**-local Delaunay graph** is defined by:

$$Del_{2,R}^{\star}(\varphi) := \bigcap_{\psi \in \Omega_{f, \emptyset_R^{\star}(\varphi)}} Del_2(\varphi \cup \psi).$$

Thus, one may give further interpretation of this subgraph: any edge of the Delaunay graph of φ possibly broken when inserting points in the $\star R$-vacuum of φ does not lie in the $\star R$-local Delaunay graph of φ. According to the notion of influence regions this leads to another way of characterizing these subgraphs $Del_{2,R}^{\star}(\varphi)$:

$$Del_{2,R}^{\star}(\varphi) = \left\{ \xi \in Del_2(\varphi) : Z_{\varphi}(\xi) \cap \emptyset_R^{\star}(\varphi) = \emptyset \right\}. \tag{4}$$

As a direct consequence of the fact that the $\boxtimes R$-vacuum of φ both includes the $\oplus R$-vacuum and the $\boxplus R$-vacuum of φ, one may assert the following proposition.

$$Del_{2,R}^{\boxtimes}(\varphi) \subset \left\{ \begin{array}{c} Del_{2,R}^{\oplus}(\varphi) \\ \text{and} \\ Del_{2,R}^{\boxplus}(\varphi) \end{array} \right\} \subset Del_{2,R}^{\bigcirc}(\varphi)$$

Let us notice that all the graphs in the above proposition are translation invariant and all of them except the $\boxplus R$-local Delaunay graph are rotation invariant. Like the original Delaunay graph (cf. (2)), the $\bigcirc R$-local Delaunay graph inherits the local property $Del_{2,R}^{\bigcirc}(\varphi_{\Delta} | \varphi_{\Delta^c}) \subset$

$\mathcal{P}_2(\overline{\mathcal{N}}_{Del^{\bigcirc}_{2,R}}(\varphi_\Delta|\varphi))$. This is untrue for the other $\star R$-local Delaunay graphs since: $Del^{\star}_{2,R}(\varphi_\Delta|\varphi_{\Delta^c}) \cap \mathcal{P}_2(\mathcal{N}_{Del^{\star}_{2,R}}(\varphi_\Delta|\varphi)) \neq Del^{\star}_{2,R}(\varphi_\Delta|\mathcal{N}_{Del^{\star}_{2,R}}(\varphi_\Delta|\varphi) \cap \Delta^c)$.

The $\bigcirc R$-local Delaunay graph behaves locally on a small scale as the Delaunay graph since, by definition, only the Delaunay edges which are not in the $\bigcirc R$-local Delaunay graph connect points close to some great enough empty space. The $\star R$-local Delaunay graphs satisfy the same property since

$$Del^{\bigcirc}_{2,R}(\varphi) \setminus Del^{\star}_{2,R}(\varphi) = \left\{\xi \in Del^{\bigcirc}_{2,R}(\varphi) : Z_\varphi(\xi) \cap \emptyset^{\star}_R(\varphi) \neq \emptyset\right\}.$$

Furthermore, the $\bigcirc R$-local Delaunay graph and the $\star R$-local Delaunay graphs coïncide more and more as R becomes greater.

Clearly, with respect to characterization (4), one may assert some further properties. The fourth property clearly expresses the expectation needed to introduce domain-wise inhibition Delaunay models.

Proposition 3 *Given any configuration φ and any Borel set Λ:*

1. $Del^{\star}_{2,R}(\varphi) = Del_2(\varphi)$ *whenever* $\emptyset^{\star}_R(\varphi) = \emptyset$.
2. $Del^{\star}_{2,R}(\varphi) \cap \mathcal{P}_2(\Lambda) = Del_2(\varphi) \cap \mathcal{P}_2(\Lambda)$ *whenever* $(\bigcup_{\xi \in Del_2(\varphi) \cap \mathcal{P}_2(\Lambda)} Z_\varphi(\xi)) \cap \emptyset^{\star}_R(\varphi) = \emptyset$.
3. $Del^{\star}_{2,R_1}(\varphi) \subset Del^{\star}_{2,R_2}(\varphi)$ *whenever* $R_1 \leq R_2$.
4. $Del^{\star}_{2,R}(\varphi_{\Lambda^c}) \subset Del^{\star}_{2,R}(\varphi)$ *uniformly on* $\varphi_\Lambda \in \Omega_\Lambda$ *whenever* $\Lambda \subset \emptyset^{\star}_R(\varphi_{\Lambda^c})$.

Proof The first two assertions are obvious. The third one is a direct consequence of $\emptyset^{\star}_{R_1}(\varphi) \supset \emptyset^{\star}_{R_2}(\varphi)$. For any $\xi \in Del^{\star}_{2,R}(\varphi_{\Lambda^c})$, one may assert that $Z_{\varphi_{\Lambda^c}}(\xi) \cap \varphi_{\Lambda^c} = \emptyset$ and $Z_{\varphi_{\Lambda^c}}(\xi) \cap \emptyset^{\star}_R(\varphi_{\Lambda^c}) = \emptyset$. As a direct consequence of $Z_\varphi(\xi) \subset Z_{\varphi_{\Lambda^c}}(\xi)$ and $\emptyset^{\star}_R(\varphi) \subset \emptyset^{\star}_R(\varphi_{\Lambda^c})$, one derives $Z_\varphi(\xi) \cap \emptyset^{\star}_R(\varphi) = \emptyset$. Finally, $\xi \in Del_2(\varphi)$ since $Z_\varphi(\xi) \cap \varphi = \emptyset$ due to $\varphi_\Lambda \subset \Lambda \subset \emptyset^{\star}_R(\varphi_{\Lambda^c})$. □

The third and last ones express our expectation. The third property is illustrated in Fig. 1 for the particular $\oplus R$-local Delaunay graph. The last property is illustrated in Fig. 2 for the particular $\oplus R$-local Delaunay graph.

2 Models

A Gibbs point process is based on some finite energy function which characterizes the system of local specifications.

2.1 Finite and Local Energy Functions and the Associated Stationary Gibbs States

In this section, we introduce the finite and local energy functions induced by the $\bullet R$-local Delaunay graph.

Definition 4 Given any fixed $R > 0$, any bounded Borel set Δ and some finite configuration φ, one defines:

- **finite $\bullet R$-energy**:

$$V^{\bullet}_R(\varphi) := \sum_{\xi \in Del^{\bullet}_{2,R}(\varphi)} \phi(\xi) \tag{5}$$

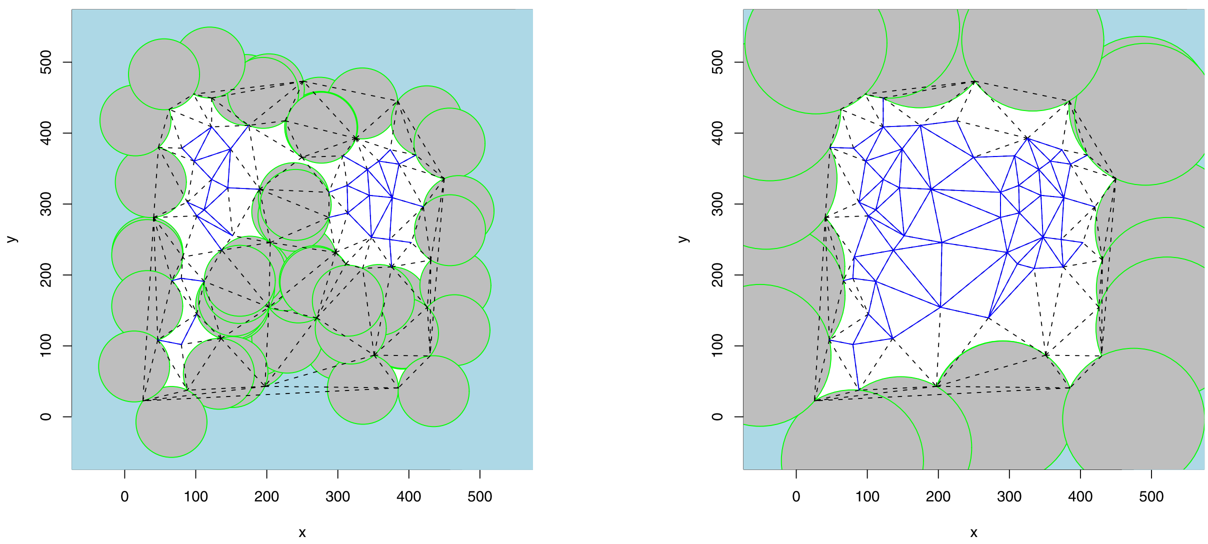

Fig. 1 Both figures represent the $\oplus R$-local Delaunay graph (*solid lines*) of the same configuration φ of some points (or vertices). In the first one $R = 50$ whereas in the second one $R = 100$. The other edges (*dotted lines*) are the residual edges of the Delaunay graph. The union of the *darker gray balls* corresponds to the quantity $\emptyset_R^{\oplus,\partial}(\varphi)$ defining in some sense the border of the $\oplus R$-vacuum of φ. The *lighter gray part* is the rest of the $\oplus R$-vacuum of φ

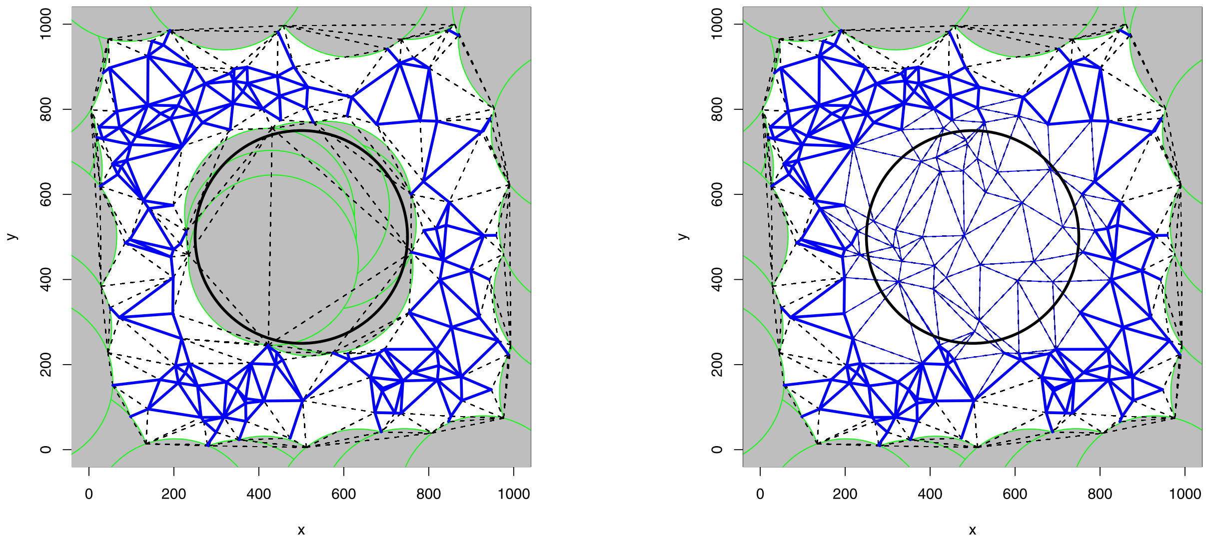

Fig. 2 The figures represent the $\oplus R$-local Delaunay graphs ($R = 200$) in *solid lines* for the configurations φ_{Λ^c} (on the *left*) and φ (on the *right*) of some points where Λ is the Borel set bounded by the *circle*. The set of edges represented graphically by wide *solid lines* corresponds to the intersection of both $Del_{2,R}^{\oplus}(\varphi_{\Lambda^c})$ and $Del_{2,R}^{\oplus}(\varphi)$ graphs. In fact, it coincides with the first graph which is included in the second one since $\varphi_{\Lambda} \subset \Lambda \subset \emptyset_R^{\oplus}(\varphi_{\Lambda^c})$. One may observe that this is obviously untrue for the Delaunay graphs with edges represented by *dotted* and *solid lines*

- **local $\bullet R$-energy**:

$$V_R^{\bullet}(\varphi_\Delta|\varphi_{\Delta^c}) := V_R^{\bullet}(\varphi) - V_R^{\bullet}(\varphi_{\Delta^c}) = \sum_{\xi \in Del_{2,R}^{\bullet}(\varphi_\Delta|\varphi_{\Delta^c})} \phi(\xi)$$

where ϕ is some pairwise interaction function.

Due to the properties of the $\star R$-local Delaunay graph, we have the following result for any $\Delta \in \mathcal{B}_R^{\star}$:

$$V_R^{\star}(\varphi_\Delta|\varphi_{\Delta^c}) = \sum_{\xi \in Del_{2,R}^{\star +}(\varphi_\Delta|\varphi_{\Delta^c})} \phi(\xi) = \sum_{\xi \in Del_{2,R}^{\star}(\varphi)\setminus Del_{2,R}^{\star}(\varphi_{\Delta^c})} \phi(\xi) \tag{6}$$

since $Del_{2,R}^{\star -}(\varphi_\Delta|\varphi_{\Delta^c}) = \emptyset$. We first recall the definition of local specifications and stationary Gibbs states based on some finite energy function V assumed to be translation invariant (like the $\star R$-energy). Let us notice that the finite energies considered in this paper except the $\boxplus R$-energy are rotation invariant.

Let us introduce π^z the Poisson process of intensity z having in particular its marginal distribution in Λ of the form:

$$\pi_\Lambda^z(F) = e^{-z|\Lambda|} \oint_\Lambda^z d\varphi \mathbb{1}_F(\varphi)$$

where F is some event in $\mathcal{F}_\Lambda$ and for any measurable function f and any positive real z:

$$\oint_\Lambda^z d\varphi f(\varphi) := \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} dx_1 \dots dx_n f(\underbrace{\{x_1, \dots, x_n\}}_{\varphi}).$$

Definition 5

1. The family of measures $\mathbf{\Pi}^V = \{\Pi_{\Lambda|\varphi^o} : \Lambda \in \mathcal{B}_b, \varphi^o \in \Omega\}$ on $(\Omega, \mathcal{F})$ related to the finite energy V is a system of local specifications:

$$\forall F \in \mathcal{F}, \quad \Pi_{\Lambda|\varphi^o}(F) := \frac{1}{Z_{\Lambda|\varphi^o}^z} \int_{\Omega_\Lambda} e^{-V_{\Lambda|\varphi^o}(\varphi)} \mathbb{1}_F(\varphi \cup \varphi_{\Lambda^c}^o) \pi_\Lambda^z(d\varphi)$$

   where $V_{\Lambda|\varphi^o}(\varphi) := V(\varphi|\varphi_{\Lambda^c}^o)$ and the partition function is given by $Z_{\Lambda|\varphi^o}^z := \int_{\Omega_\Lambda} e^{-V_{\Lambda|\varphi^o}(\varphi)} \pi_\Lambda^z(d\varphi)$.
2. $\mathcal{G}_0(V) = \mathcal{G}_0(\mathbf{\Pi}^V)$ is the set of stationary Gibbs states defined as some probability measure μ such that $\mu(F|\mathcal{F}_{\Lambda^c}) = \Pi_{\Lambda|\cdot}(F)$ μ-a.s. for all $F \in \mathcal{F}$ and $\Lambda \in \mathcal{B}$.

One first checks that the partition function $Z_{\Lambda|\varphi^o}^z$ is finite. For any bounded Borel set Δ and any family $G = \{G(\varphi)\}_{\varphi \in \Omega}$ of graphs, let us introduce $\mathcal{N}_G(\Delta|\varphi) := \bigcup_{\psi \in \Omega_\Delta} (\mathcal{N}_G(\psi|\psi \cup \varphi_{\Delta^c}) \cap \mathcal{P}(\varphi_{\Delta^c}))$. For convenience, $\varphi^o_{\mathcal{N}_{Del_{2,R}^{\star}}(\Lambda|\varphi^o)}$ is simply denoted by $\widetilde{\varphi}_{\Lambda^c}^o$. Due to the local property of the Delaunay graph, $\widetilde{\varphi}_{\Lambda^c}^o$ is a finite configuration. Furthermore, since $V(\varphi|\varphi_{\Lambda^c}^o) = V(\varphi|\widetilde{\varphi}_{\Lambda^c}^o) = V(\varphi \cup \widetilde{\varphi}_{\Lambda^c}^o) - V(\widetilde{\varphi}_{\Lambda^c}^o)$ the partition function $Z_{\Lambda|\varphi^o}^z = e^{V(\widetilde{\varphi}_{\Lambda^c}^o)} \int_{\Omega_\Lambda} e^{-V(\varphi \cup \widetilde{\varphi}_{\Lambda^c}^o)} \pi_\Lambda^z(d\varphi)$ is finite whenever $V(\widetilde{\varphi}_{\Lambda^c}^o) < +\infty$ and V is stable. In the context of this paper, this is valid when the range of the pairwise interaction function is included in $\mathbb{R}^+$.

More generally, let us notice that, in the particular dimensional case of $d = 2$, stability occurs when the pairwise interaction function is lowerbounded due to the linear complexity of the Delaunay graph.

2.2 Properties of the Local $\bullet R$-Energy

All the $\bullet R$-local Delaunay graphs do not contain edges larger than $2R$. One may then expect that the local $\bullet R$-energy inherits the local property.

Proposition 4 *For any $\Delta \in \mathcal{B}_b$, there exists $\Lambda^\bullet \in \mathcal{B}_b$ such that*

$$V_R^\bullet(\varphi_\Delta|\varphi_{\Delta^c}) = V_R^\bullet(\varphi_\Delta|\varphi_{\Lambda^\bullet\setminus\Delta}). \tag{7}$$

In particular, one may choose $\Lambda^\bigcirc = \Delta \oplus 2R$ and $\Lambda^\star = \Delta \oplus 6R$.

Proof $Del_2(\varphi_\Delta|\varphi_{\Delta^c}) \subset \mathcal{P}_2(\mathcal{N}_{Del}(\varphi_\Delta|\varphi))$ where $\mathcal{N}_{Del}(\varphi_\Delta|\varphi)$ denotes the set of Delaunay neighbours of the points of φ_Δ inside the whole configuration φ. It is then direct to assert that $Del_{2,R}^\bigcirc(\varphi_\Delta|\varphi_{\Delta^c}) \subset \Delta \oplus 2R$ and consequently $Del_{2,R}^\bigcirc(\varphi_\Delta|\varphi_{\Delta^c}) = Del_{2,R}^\bigcirc(\varphi_\Delta|\varphi_{(\Delta\oplus 2R)\setminus\Delta})$.

The other $\star$ cases standing for $\oplus$, $\boxplus$ and $\boxtimes$ still remain to be dealt with. The first step is to localize the residual edges of $Del_{2,R}^\star(\varphi_\Delta|\varphi_{\Delta^c})$. Such edges ξ are of two kinds depending on whether they satisfy $Z_\varphi(\xi) \neq Z_{\varphi_{\Delta^c}}(\xi)$ or not. An edge of the first kind necessarily belongs to $Del_{2,R}^\bigcirc(\varphi_\Delta|\varphi_{\Delta^c}) = Del_{2,R}^\bigcirc(\varphi_\Delta|\varphi_{(\Delta\oplus 2R)\setminus\Delta})$. Alternatively, an edge ξ of the second type has to be such that $\emptyset_R^\oplus(\varphi_{\Delta^c}) \cap Z_\varphi(\xi) \neq \emptyset$. In such a case, there exists a ball of radius R both intersecting $Z_\varphi(\xi)$ and φ_Δ. This leads to $Del_{2,R}^\star(\varphi_\Delta|\varphi_{\Delta^c}) \subset \mathcal{P}_2(\Delta \oplus 4R)$. Let us notice that this result is valid for any φ and so for $\varphi_{\Delta\oplus 6R}$, that is, $Del_{2,R}^\star(\varphi_\Delta|\varphi_{(\Delta\oplus 6R)\setminus\Delta}) \subset \mathcal{P}_2(\Delta \oplus 4R)$. The last step is to prove that $Del_{2,R}^\star(\varphi_\Delta|\varphi_{\Delta^c}) \cap \mathcal{P}_2(\Delta \oplus 4R) = Del_{2,R}^\star(\varphi_\Delta|\varphi_{(\Delta\oplus 6R)\setminus\Delta}) \cap \mathcal{P}_2(\Delta \oplus 4R)$. Since this result is direct when ξ is of the first kind, one concentrates on the edges ξ of the second kind. To negate the result, one needs to find some ball of radius R both intersecting $Z_\varphi(\xi)$ and $(\Delta \oplus 6R)^c$ which is clearly impossible. Finally, in order to establish the proof, one has:

$$\begin{aligned} Del_{2,R}^\star(\varphi_\Delta|\varphi_{\Delta^c}) &= Del_{2,R}^\star(\varphi_\Delta|\varphi_{\Delta^c}) \cap \mathcal{P}_2(\Delta \oplus 4R) \\ &= Del_{2,R}^\star(\varphi_\Delta|\varphi_{(\Delta\oplus 6R)\setminus\Delta}) \cap \mathcal{P}_2(\Delta \oplus 4R) \\ &= Del_{2,R}^\star(\varphi_\Delta|\varphi_{(\Delta\oplus 6R)\setminus\Delta}). \end{aligned}$$ □

Since the $\bullet R$-local Delaunay graph can have the same behavior as the Delaunay graph, the associated local $\bullet R$-energy is unfortunately not stable. However, one may provide a new property for the $\star R$-energy which is an extension of the local stability.

Definition 6 Let $\widetilde{\mathcal{B}}$ be a cofinal set of $\mathcal{B}$, that is, each $\Delta \in \mathcal{B}$ is contained in some $\widetilde{\Delta} \in \widetilde{\mathcal{B}}$. An energy function V is said to be $\widetilde{\mathcal{B}}$-cofinally locally stable if it satisfies the $\widetilde{\mathcal{B}}$-cofinally local stability property:

($\widetilde{\mathcal{B}}$-CLS) there exists $K \geq 0$ such that for any $\varphi \in \Omega$ and $\widetilde{\Delta} \in \widetilde{\mathcal{B}}$:

$$V\left(\varphi_{\widetilde{\Delta}}|\varphi_{\widetilde{\Delta}^c}\right) \geq -K\#(\varphi_{\widetilde{\Delta}}). \tag{8}$$

In the particular case when $K = 0$, V satisfies the $\widetilde{\mathcal{B}}$-cofinally inhibition property.

The classical local stability is then equivalent to the $\mathcal{B}$-cofinally local stability.

Proposition 5 *The $\star R$-energy $V_R^\star$ with a nonnegative interaction function satisfies the $\mathcal{B}_R^\star$-cofinally inhibition property.*

Proof This is a direct consequence of property (6) with $K = 0$. □

2.3 Correlation Function

Given any outside configuration φ^o, Ruelle [14] has introduced the following quantity:

$$\rho^z_{\Delta,\Lambda|\varphi^o}(\varphi) := \frac{z^{\#\varphi}}{Z^z_{\Lambda|\varphi^o}} \oint^z_{\Lambda\setminus\Delta} d\psi e^{-V(\varphi\cup\psi|\varphi^o_{\Lambda^c})}. \tag{9}$$

As a particular case, $\Delta = \emptyset$, one can derive the correlation function $\rho^z_{\Lambda|\varphi^o}(\varphi) := \rho^z_{\emptyset,\Lambda|\varphi^o}(\varphi)$, satisfying

$$\rho^z_{\Delta,\Lambda|\varphi^o}(\varphi) \le \rho^z_{\Lambda|\varphi^o}(\varphi)\,.$$

A well-known property is that this correlation function is upperbounded by the correlation function of some stationary Poisson process. For some cofinal set $\widetilde{\mathcal{B}}$ of $\mathcal{B}$, let us define the following property:

($\widetilde{\mathcal{B}}$-UC) for any $\Delta \in \mathcal{B}_b$, there exists $\zeta > 0$ such that for any $\varphi \in \Omega_f$,

$$\rho^z_{\Delta,\Lambda|\varphi^o}(\varphi) \le \zeta^{\#\varphi}$$

uniformly on $\varphi^o \in \Omega$ and $\Lambda \in \widetilde{\mathcal{B}}$.

For any $\Delta \in \mathcal{B}_b$ and any cofinal set $\widetilde{\mathcal{B}}$ of $\mathcal{B}$, let us introduce the subset $\widetilde{\mathcal{B}}_\Delta$ of $\widetilde{\mathcal{B}}$ defined as

$$\widetilde{\mathcal{B}}_\Delta = \left\{ \widetilde{\widetilde{\Delta}} \in \widetilde{\mathcal{B}} : |\widetilde{\widetilde{\Delta}} \setminus \Delta| = \min_{\substack{\widetilde{\Delta}\in\widetilde{\mathcal{B}} \\ \widetilde{\Delta}\supset\Delta}} |\widetilde{\Delta} \setminus \Delta| \right\}$$

Proposition 6 *If some energy function V is $\widetilde{\mathcal{B}}$-cofinally locally stable then for any $\Delta \in \mathcal{B}_b$, $\varphi \in \Omega_\Delta$, $\varphi^o \in \Omega$ and $\Lambda \in \mathcal{B}_b$ such that $\Lambda \supset \widetilde{\Delta}$ with $\widetilde{\Delta} \in \widetilde{\mathcal{B}}_\Delta$:*

$$\rho^z_{\Delta,\Lambda|\varphi^o}(\varphi) \le e^{ze^K|\widetilde{\Delta}\setminus\Delta|}(ze^K)^{\#\varphi} \le (ze^K e^{ze^K|\widetilde{\Delta}\setminus\Delta|})^{\#\varphi}.$$

In other words, **(**$\widetilde{\mathcal{B}}$-***UC*****)** *is a consequence of* **(**$\widetilde{\mathcal{B}}$-***CLS*****)**.

Proof

$$\begin{aligned}
\rho^z_{\Lambda|\varphi^o}(\varphi) &= \frac{z^{\#\varphi}}{Z^z_{\Lambda|\varphi^o}} \oint^z_{\Lambda\setminus\Delta} d\psi\, e^{-V(\varphi\cup\psi|\varphi^o)} \\
&= \frac{z^{\#\varphi}}{Z^z_{\Lambda|\varphi^o}} \oint^z_{\widetilde{\Delta}\setminus\Delta} d\psi_1 \oint^z_{\Lambda\setminus\widetilde{\Delta}} d\psi_2\, e^{-V(\varphi\cup\psi_1|\psi_2\cup\varphi^o)-V(\psi_2|\varphi^o)} \\
&\le \frac{(ze^K)^{\#\varphi}}{Z^z_{\Lambda|\varphi^o}} \oint^z_{\widetilde{\Delta}\setminus\Delta} d\psi_1 e^{K\#\psi_1} \oint^z_{\Lambda\setminus\widetilde{\Delta}} d\psi_2\, e^{-V(\psi_2|\varphi^o)} \quad \text{(by } (\widetilde{\mathcal{B}}\textbf{-CLS}))
\end{aligned}$$

$$\leq \left(ze^K\right)^{\#\varphi} e^{ze^K|\widetilde{\Delta}\setminus\Delta|} \frac{Z^z_{\Lambda\setminus\widetilde{\Delta}|\varphi^o}}{Z^z_{\Lambda|\varphi^o}}$$

$$\leq \left(ze^K\right)^{\#\varphi} e^{ze^K|\widetilde{\Delta}\setminus\Delta|} \quad (\text{since } Z^z_{\Lambda\setminus\widetilde{\Delta}|\varphi^o} \leq Z^z_{\Lambda|\varphi^o}) \qquad \square$$

In the particular case when V is $\mathcal{B}^\star_R$-cofinally locally stable, one obtains:

$$\rho^z_{\Lambda|\varphi^o}(\varphi) \leq \left(ze^K e^{ze^K|B(0,R)|}\right)^{\#\varphi}$$

since $|\varphi \oplus R|$ is then upperbounded by $\#\varphi \times |B(0,R)|$.

The correlation functions $\rho^z_{\Delta,\Lambda|\varphi^o}(\cdot)$, defined in (9), play an important role in the expression of the Radon-Nikodym derivative of the local specification $\Pi_{\Lambda|\varphi^o}$ with respect to some Poisson process in Δ. In particular, the following probabilities $\Pi_{\Lambda|\varphi^o}(F_\Delta)$ for any bounded Borel set $\Delta \subset \Lambda$, and any $F_\Delta \in \widetilde{\mathcal{F}}_\Delta$, are controlled uniformly on $\Lambda \supset \Delta$ and φ^o:

$$\begin{aligned}
\Pi_{\Lambda|\varphi^o}(F_\Delta) &= \frac{1}{Z^z_{\Lambda|\varphi^o}} \int_{\Omega_\Lambda} e^{-V(\varphi|\varphi^o_{\Lambda^c})} \mathbb{1}_{F^{loc}_\Delta}(\varphi) \pi^z_\Lambda(d\varphi) \\
&= \oint_\Delta^z d\varphi^i \mathbb{1}_{F^{loc}_\Delta}(\varphi^i) \left(\frac{1}{Z^z_{\Lambda|\varphi^o}} \oint^z_{\Lambda\setminus\Delta} d\varphi^b e^{-V(\varphi^i\cup\varphi^b|\varphi^o_{\Lambda^c})} \right) \\
&= \oint_\Delta^z d\varphi^i \mathbb{1}_{F^{loc}_\Delta}(\varphi^i) \frac{\rho^z_{\Delta,\Lambda|\varphi^o}(\varphi^i)}{z^{\#\varphi^i}} \qquad (10)
\end{aligned}$$

where F^{loc}_Δ denotes the projection of F_Δ onto $\mathcal{F}_\Delta$.

Consequently, under the $\widetilde{\mathcal{B}}$-cofinally local stability of V, one derives for any bounded Borel sets $\Delta \subset \Lambda$, and any $F_\Delta \in \widetilde{\mathcal{F}}_\Delta$ that:

$$\begin{aligned}
\Pi_{\Lambda|\varphi^o}(F_\Delta) &= \oint_\Delta^z d\varphi \mathbb{1}_{F^{loc}_\Delta}(\varphi) \frac{\rho^z_{\Delta,\Lambda|\varphi^o}(\varphi)}{z^{\#\varphi}} \\
&\leq \oint_\Delta^z d\varphi \mathbb{1}_{F^{loc}_\Delta}(\varphi) \frac{(ze^K e^{ze^K|\widetilde{\Delta}\setminus\Delta|})^{\#\varphi}}{z^{\#\varphi}} \\
&= \oint_\Delta^{ze^K e^{ze^K|\widetilde{\Delta}\setminus\Delta|}} d\varphi \mathbb{1}_{F^{loc}_\Delta}(\varphi) \qquad (11)
\end{aligned}$$

In particular, for the event $F_\Delta = [N_\Delta \geq m]$ one obtains:

$$\Pi_{\Lambda|\varphi^o}([N_\Delta \geq m]) = \sum_{k=m}^{+\infty} \frac{z^k}{k!} \int_{\Delta^k} dx_1 \ldots dx_k \rho^z_{\Delta,\Lambda|\varphi^o}(\{x_1,\ldots,x_m\}) \leq \sum_{k=m}^{+\infty} \frac{(ze^K e^{ze^K|\widetilde{\Delta}\setminus\Delta|}|\Delta|)^k}{k!}$$

which does not depend on Λ and φ^o.

3 Existence of Inhibition Stationary Gibbs States

In this paper we present and use the tool introduced by Georgii-Häggström [8] giving sufficient conditions in order to provide the existence of stationary Gibbs states. Let us point out that without this tool, we could not deal with the inhibition pairwise Gibbs process based on the $\bigcirc R$-local Delaunay graph. We will discuss this further on. We simplify the presentation of the result proposed in [8] mainly for two reasons. The first one is that we only concentrate on Gibbs models with local energy functions satisfying the following local property:

(**L**): for any $\Lambda \in \mathcal{B}_b$ there exists some $\widetilde{\Lambda} \in \mathcal{B}_b$ such that for all $\varphi \in \Omega$:

$$V(\varphi_\Lambda|\varphi_{\Lambda^c}) = V(\varphi_\Lambda|\varphi_{\widetilde{\Lambda}\setminus\Lambda}). \tag{12}$$

The second one is that we are here only interested in the extension of the particular case considered in [8] of the one-type particle process with nonnegative pairwise interaction based on the complete graph. The proof proposed here is a direct adaptation replacing the complete graph with the $\bullet R$-local Delaunay graph. However, let us notice that this is only possible because this tool allows us to choose some arbitrary fixed boundary φ^o (for instance, the free boundary $\emptyset$). The main assumption of the tool is the following one:

(**IM**): there exists $\varphi^o \in \Omega$ such that

$$I(\Pi_{\Lambda|\varphi^o}; \pi^z_\Lambda) \le c|\Lambda| \tag{13}$$

for some constant $c \in [0, +\infty[$

where $I(P; Q)$ denotes the relative entropy (or Kullback-Leibler information) of two probability measures P and Q on the same measurable space. Let us point out that this assumption is satisfied whenever the following one is satisfied

(**GE**): there exists $\varphi^o \in \Omega$ such that for any $\varphi \in \Omega_\Lambda$

$$V_{\Lambda|\varphi^o}(\varphi) := V(\varphi|\varphi^o_{\Lambda^c}) \ge -c_0|\Lambda| \tag{14}$$

for some constant $c_0 \in [0, +\infty[$.

Indeed, as a direct consequence of $Z^z_{\Lambda|\varphi^o} \ge \pi^z(\{\emptyset\}) = e^{-z|\Lambda|}$, one derives

$$I(\Pi_{\Lambda|\varphi^o}; \pi^z_\Lambda) = -\int V(\varphi_\Lambda|\varphi^o_{\Lambda^c})\Pi_{\Lambda|\varphi^o}(d\varphi) - \log Z^z_{\Lambda|\varphi^o} \le \underbrace{(c_0 + z)}_{c}|\Lambda|.$$

The result is then proposed as a lemma and the proof directly extracted from [8] is given mainly for convenience.

Lemma 7 *If V is a finite energy function such that* (***IM***) *and* (**L**) *then $\mathcal{G}_0(V)$ is nonempty.*

Proof Given any $n \in \mathbb{Z}^d$, let us introduce $\Lambda_n = [-n - 1/2, n + 1/2[^d$, $L_n = \Lambda_n \cap \mathbb{Z}^d$, $v_n = (2n+1)^d = \#L_n = |\Lambda_n|$ and Π_n the projection of $\Pi_{\Lambda_n|\varphi^o}$ on Λ_n. Let $\mathcal{L}$ denote the class of all measurable functions f which are local and tame, that is, there exists some $l \ge 1$ such that $f = f(\cdot \cap \Lambda_l)$ and $|f| \le b(1 + N_{\Lambda_l})$ for some constant $b = b(f) < +\infty$. The topology $\mathcal{T}_{\mathcal{L}}$ of the local convergence on the set of all probability measures P with $\int N_\Delta dP < +\infty$ for each finite box Δ is defined as the weak topology induced by $\mathcal{L}$. One defines the relative entropy density $I(P)$ of any $(\vartheta_i)_{i\in\mathbb{Z}^d}$-invariant probability measure P as

$$I(P) = \lim_{k\to+\infty} v_k^{-1} I(P_{\Lambda_k}; \pi^z_{\Lambda_k})$$

where P_{Λ_k} denotes respectively the projections of P onto Ω_{Λ_k}.
The proof is decomposed into two main stages.

Existence of an Accumulation Point Π Let us consider the following sequence of probability measures $(\overline{\Pi}_n)$ defined by

$$\overline{\Pi}_n = v_n^{-1} \sum_{i \in L(n)} \Pi_n \circ \vartheta_i^{-1}.$$

In particular, Π_n is not $(\vartheta_i)_{i\in\mathbb{Z}^d}$-invariant probability measure. However, Georgii-Häggström have proved that a subsequence of $(\overline{\Pi}_n)$ converges to some $(\vartheta_i)_{i\in\mathbb{Z}^d}$-invariant probability measure Π. The rest of this step is devoted to this result. Let us start by introducing $\widehat{\Pi}_n$ the probability measure in $\mathbb{R}^d$ such that its marginal distributions on the disjoint blocks $\Lambda_n + (2n+1)i$, $i \in \mathbb{Z}^d$ are independent with identical distribution Π_n and the $(\vartheta_i)_{i\in\mathbb{Z}^d}$-invariant associated probability measure

$$\widetilde{\Pi}_n = v_n^{-1} \sum_{i \in L(n)} \widehat{\Pi}_n \circ \vartheta_i^{-1}.$$

By arguing the fact that the sublevel sets $\{I \le c\}$ are sequentially compact in the topology $\mathcal{T}_{\mathcal{L}}$, one may derive that the sequence $(\widetilde{\Pi}_n)$ has a subsequential limit Π in $\mathcal{T}_{\mathcal{L}}$ as a direct consequence of $I(\widetilde{\Pi}_n) = v_n^{-1} I(\Pi_n; \pi^z_{\Lambda_n})$ and **(IM)**. The expected result derives from the fact that the authors proved, with a slight variant of Lemma 5.7 [10], that the sequences $(\overline{\Pi}_n)$ and $(\widetilde{\Pi}_n)$ are asymptotically equivalent in the topology $\mathcal{T}_{\mathcal{L}}$.

Verification of the Gibbs Property For any given local and bounded function f and any bounded Borel set Δ, let us denote by f_Δ the function defined as $f_\Delta(\varphi) = \int f d\Pi_{\Delta|\varphi}$. As in [8], the proof is only given for $\Delta = \Lambda_0$ for notational convenience. By noticing that for any $x \in \mathbb{R}^d$, $f_\Delta \circ \vartheta_x = (f \circ \vartheta_x)_{\vartheta_x^{-1}(\Delta)}$, the equilibrium equation on $\overline{\Pi}_n$ is then directly provided since for any integer n

$$\begin{aligned}
\int f_\Delta d\overline{\Pi}_n &= v_n^{-1} \sum_{i \in L_n} \int f_\Delta \circ \vartheta_i d\Pi_n \\
&= v_n^{-1} \sum_{i \in L_n} \int (f \circ \vartheta_i)_{\vartheta_i^{-1}(\Delta)} d\Pi_n \\
&= v_n^{-1} \sum_{i \in L_n} \int f \circ \vartheta_i d\Pi_n \\
&= \int f d\overline{\Pi}_n.
\end{aligned}$$

In the simple case where f is a local and bounded function which implies that f_Δ is a local and bounded function too, one directly derives the DLR equation for Π:

$$\int f_\Delta d\Pi = \int f d\Pi.$$

Finally, $\Pi^0 = \int_{\Lambda_0} \Pi \circ \vartheta_x^{-1} dx$ is a $(\vartheta_x)_{x\in\mathbb{R}^d}$-invariant probability measure satisfying the DLR equation since

$$\int f_\Delta d\Pi^0 = \int_{\Lambda_0} \int f_\Delta \circ \vartheta_x d\Pi dx$$

$$= \int_{\Lambda_0} \int (f \circ \vartheta_x)_{\vartheta_x^{-1}(\Delta)} d\Pi dx$$
$$= \int_{\Lambda_0} \int f \circ \vartheta_x d\Pi dx$$
$$= \int f d\Pi^0. \qquad \square$$

The main result of this paper is then a direct consequence of the previous lemma.

Theorem 8 $\mathcal{G}_0(V_R^\bullet)$ *is nonempty whenever the translation invariant pairwise interaction function is nonnegative.*

Proof We already know from Proposition 4 that **(L)** is satisfied. Condition **(GE)** is ensured by the nonnegativeness of the pairwise interaction function and by choosing $\varphi^o = \emptyset$ (i.e. free boundary). $\square$

Before commenting on assumptions **(IM)** and more particularly **(GE)**, let us recall some more usual tools related to the existence of stationary Gibbs states. Preston [13] dedicated a large part of his book to proposing such a tool. In [3], we proposed some simpler sufficient conditions based on the local energy in order to satisfy the assumptions of Preston's theorem [13, Theorem 4.3, p. 58]. In addition to **(L)** (or more generally, as in [3], the quasilocality of the local energy), one needs the local stability property which may be written as follows:

(**LS**) there exists $K \geq 0$ such that for any $\varphi \in \Omega$ and $\Delta \in \mathcal{B}_b$:

$$V(\varphi_\Delta | \varphi_{\Delta^c}) \geq -K\#(\varphi_\Delta). \tag{15}$$

Even if **(LS)** (or more generally **($\widetilde{\mathbf{B}}$-CLS)**) and **(GE)** seem to be quite similar in their energetical expressions, they are fundamentally different mainly due to the fact that the constant K in **(LS)** is uniformly independent of every boundary configuration whereas the constant c_0 in **(GE)** depends on the fixed boundary configuration φ^o. The first one is then much more restrictive than the second one.

In the first version of this paper, we did not realize the power of the tool proposed by Georgii-Häggström [8] and since we are used to manipulating strong assumptions similar to **(LS)**, we restricted ourselves to investigating Gibbs models with pairwise interactions based on the $\star R$-local Delaunay graph. This graph was initially tailor-made in order to satisfy the extended property **($\mathcal{B}_R^\star$-CLS)** of **(LS)** which provides the condition **(3-10)** required in Preston's theorem 4.3 [13]. More precisely, this condition **(3-10)** in [13] is a direct consequence of (11) provided by assumption **($\widetilde{\mathcal{B}}$-UC)** and more particularly by **($\mathcal{B}_R^\star$-UC)**. However, one may expect that stronger assumptions provide further properties on the associated stationary Gibbs states. Indeed, **($\widetilde{\mathcal{B}}$-UC)** additionally ensures that any existing stationary Gibbs state Π satisfies an equation similar to (11) and inherits the property of finiteness of the moments of N_Δ (for any bounded Borel set Δ). This is expressed by the following proposition (given without proof) which is directly applicable to the finite energy $V_R^\star$.

Proposition 9 *By assuming that V is some translation invariant energy function satisfying* **($\widetilde{\mathcal{B}}$-UC)** *(implied by* **($\widetilde{\mathcal{B}}$-CLS)**) *and* **(L)**, *one may assert that*:

1. $\mathcal{G}_0(V)$ *is nonempty.*

2. *For any* $\Pi \in \mathcal{G}_0(V)$ *and* $\Delta \in \mathcal{B}_b$

$$\Pi([N_\Delta \geq m]) \leq \sum_{k=m}^{+\infty} \frac{(\xi|\Delta|)^k}{k!} \tag{16}$$

and for any $k \in \mathbb{N}$

$$\Pi(N_{\Delta,k}) \leq e^{\xi|\Delta|} \times (\xi|\Delta|)^k \quad \textit{and} \quad \Pi(e^{N_\Delta}) \leq e^{e\xi|\Delta|} \tag{17}$$

where ξ *is related to* **($\widetilde{\mathcal{B}}$-UC)** *and the function* $N_{\Delta,k}$ *is defined for any configuration* φ *by* $N_{\Delta,k}(\varphi) = (N_\Delta(\varphi))!/(N_\Delta(\varphi) - k)!$.

We finally end this section with some concluding remarks.

Remark 1 The previous model could be easily extended by adding interaction terms of all order whenever the related interaction functions are nonnegative. A second point is that this work could be adapted to some Delaunay subgraphs such as the Gabriel graph and the Relative neighbour graph.

Remark 2 In the plane, the maximum number of Voronoi vertices (or equivalently, the number of Delaunay triangles) is an upperbound for the number of holes (or more precisely, the Euler characteristic) generated by the Quermass-interaction model studied in [11]. Thus, there is a strong link between our model and the Quermass-interaction model in the planar case when the grains are disks of fixed radii. The Quermass-interaction model [11] and nearest-neighbour models [1] defined thanks to the Delaunay graph raised problems of stability in dimensions greater than two whereas models presented here work in any dimension.

Remark 3 The stability of the finite energy and the temperedness of the mutual energy (see [14, p. 32]), implied by the finite range property, provide results concerning the existence of the pressure with free boundary condition and thermodynamic limits of microcanonical, canonical and grand canonical ensembles [14, pp. 41–58].

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